WORKING PAPER

DANMARKS NATIONALBANK

9 MARCH 2023 — NO. 193

Drivers of Real Interest Rates in A Two-country, General-equilibrium, OLG Model

Frederik Bjørn Christensen
FBC.eco@cbs.dk

COPENHAGEN BUSINESS SCHOOL

Jesper Pedersen
jep@formuepleje.dk

FORMUEPLEJE

WORKING PAPER — DANMARKS NATIONALBANK
9 MARCH 2023 — NO. 193

Drivers of Real Interest Rates in A Two-country, General-equilibrium, OLG Model

Abstract

What can explain the long-term decline in equilibrium real interest rates? We analyze the importance of three of the most cited drivers; decreasing fertility, decreasing mortality, and a slowdown of technological growth. We do this through the lens of a general-equilibrium, two-country, overlapping generations model with international capital markets and trade in goods. Using the US as a proxy for the world economy, we find that all three factors put downward pressure on the global real interest rate. The model predicts a 2.25 percentage point decrease from 1950 until today with falling mortality generating most of the decline. We calibrate the second country on Danish data and show how differences in mortality can help explain the build-up of the large Danish net foreign asset position. Our results suggest that the secular decline in real interest rates is not fully over; the drivers considered in this analysis are likely to depress real interest rates by another 0.25 percentage points going forward from today until 2050.

Resume

Hvad kan forklare det langsigtede fald i ligevægtsrealrenter? Vi analyserer tre af de mest citerede årsager: faldende fertilitet, faldende dødelighed og en opbremsning af teknologisk vækst. Analysen foretages i en overlappende generationsmodel for to lande og med generel ligevægt, hvor der er anvendt internationale kapitalmarkeder og handel med varer. Ved at bruge USA som en proxy for verdensøkonomien viser vi først, at alle tre faktorer lægger et nedadgående pres på realrenten. Modellen peger på et fald på 2,25 procentpoint fra 1950 og indtil i dag, hvor faldende dødelighed er den vigtigste drivkraft bag faldet. Vi kalibrerer det andet land på danske data og viser, hvordan forskelle i dødelighed kan bidrage til at forklare opbygningen af den store danske nettoformue over for udlandet. Vores resultater peger på, at det sekulære fald i realrenterne ikke er forbi; de faktorer, der tages i betragtning i denne analyse, vil sandsynligvis reducere realrenten med yderligere 0,25 procentpoint fra niveauet i dag og frem mod 2050.

Drivers of Real Interest Rates in a Two-country, General-equilibrium, OLG Model*

Frederik Bjørn Christensen† and Jesper Pedersen‡

January 2023

Abstract

What can explain the long-term decline in equilibrium real interest rates? We analyze the importance of three of the most cited drivers; decreasing fertility, decreasing mortality, and a slowdown of technological growth. We do this through the lens of a general-equilibrium, two-country, overlapping generations model with trade in goods and capital. Using the US as a proxy for the world economy, we first find that all three factors put downward pressure on the global real interest rate. The model predicts a 2.25 percentage point decrease from 1950 until today with falling mortality generating most of the decline. We then calibrate the second country on Danish data and show how differences in mortality can help explain the build-up of the large Danish net foreign asset position. Our results suggest that the secular decline in real interest rates is not fully over; the drivers that we consider are likely to depress real interest rates by another 0.25 percentage points toward 2050.

JEL classification: E21, E22, E37, E43, E47

Keywords: Real Interest Rates, Mortality, Fertility, Technology, Trade, International Capital Markets

---

*The authors wish to thank colleagues from Danmarks Nationalbank for their valuable comments. The views expressed in this paper are those of the authors and do not necessarily correspond to those of Danmarks Nationalbank. This project was undertaken while both authors were employed by Danmarks Nationalbank. The results in this Working Paper neither reflect the policy position of Danmarks Nationalbank nor of Formuepleje.

†Copenhagen Business School, fbc.eco@cbs.dk

‡Formuepleje, jep@formuepleje.dk.

The real interest rate has been falling for decades in many economies.[1] This is also the case in Denmark.[2]. The natural real interest rate is a theoretical concept and can be defined as the rate of interest that balances desired savings and investment in an economy where output equals potential output and prices are stable. Although unobservable, it is one of the most important variables for central banks, as it determines the monetary policy position, i.e., whether monetary policy stimulates or dampens growth. Thus, understanding the driving forces of the real natural interest rate is of the utmost importance for central banks.[3]

The economic literature has pointed to many possible drivers of the decline in real interest rates.[4] Overall, the decline reflects that desired savings on a global scale have increased relative to desired investments. Hence, everything that affects savings positively and/or investments negatively can potentially explain the fall in the real interest rate.

One explanation could be the decline in total factor productivity (TFP) growth that has been observed in many advanced economies since the 1990s. This may reduce the incentive for firms to invest in new equipment. At the same time, slower TFP growth and hence slower growth in earnings may have contributed to higher household savings rates. If a household expects its income to increase substantially in the future, it has a smaller incentive to save. From that logic, weaker growth in income tends to increase savings, putting downward pressure on interest rates.

Another prominent explanation is changing demographics, see e.g. Bielecki et al. (2020) and Lisack et al. (2017). In particular, increasing longevity and decreasing fertility lead to an aging population which again drives up desired savings and pushes down real interest rates. The role of demographics is important for two reasons. First, most advanced economies are going through a demographic transition, and a deeper understanding of the implications is called for. Second, while there is great uncertainty associated with all projections, the uncertainty is perhaps the smallest with respect to demographics. Hence, if we build a model that fits key properties of equilibrium interest rates in the sample, it is reasonable to assume that feeding the model with projections for future demographic developments provides decent projections for future interest rates.

The equilibrium interest rate must be estimated using statistical methods or derived through theoretical models, as it is an unobserved, theoretical object. In this paper, we contribute to the literature on drivers of real interest rates by studying the evolution of equilibrium interest rates through the lens of a computable, general-equilibrium, two-country, overlapping generations model with international capital markets and trade in goods. Using

[1]See e.g. Adolfsen and Pedersen (2019) for a discussion and the references therein

[2]See Figure 7

[3]For a discussion of $r_t^\star$ and its role for the conduct of monetary policy, see e.g. Adolfsen and Pedersen (2019). A low $r_t^\star$ has been a motivating factor behind the monetary policy strategy reviews both at the ECB and in the FED, see e.g. www.ecb.europa.eu/home/search/review and https://www.federalreserve.gov/monetarypolicy/review-of-monetary-policy-strategy-tools-and-communications.htm.

[4]The literature on this topic is vast. Brand et al. (2018) provide an extensive overview of drivers and models for the estimation of equilibrium rates. Likewise, Teulings and Baldwin (2014) have a more global view of the drivers of equilibrium rates.

this model, we specifically evaluate three of the most cited drivers for decreasing real interest rates; decreasing fertility, decreasing mortality, and a slowdown of technological growth. In doing so, we also consider what will happen to the real interest rate if current trends persist. In the analysis, we consider the impact of these drivers on real interest rates both in the long-run steady state and in the short-run transition path. Moreover, we conduct experiments to decompose the total effect into contributions from each driver in isolation. On every occasion, we model changes in technology, fertility, and mortality in accordance with empirical evidence as MIT shocks (shocks in a world without shocks). In a first exercise, we consider a one-country setting by calibrating both model countries as one, taking the US as a proxy for the world economy. We find that all three drivers exert downward pressure on the real interest rate with falling mortality capturing most of the decline. In total, the real interest rate decreases by around 2.4 percentage points over a 100-year period from 1950 to 2050. Our results suggest that the secular decline in real interest rates is not fully over. Of the total decline, 0.25 percentage points are likely to transpire from the present until 2050. This is, of course, under the assumption that the driving forces continue their trends. In a second exercise, we calibrate a second country on Danish data while still using the US as a proxy for the broader world economy. This allows us to study international capital flows and how differences in the underlying forces and desire for savings across countries may help explain differences in net foreign asset positions. This is of particular interest for Denmark, which has posted substantial current account surpluses (likely to reach around 10 percent of GDP in 2022) for a sustained period of time, which has led to the build-up of large net foreign assets vis-à-vis the world economy. Although we focus on Denmark, the model is versatile and could be used to study the interaction between any two regions, i.e., the US vs. the Euro Area or the US vs. China.

It is useful to clarify our definition of the equilibrium interest rate to better interpret and understand the results of this paper.[5] In DSGE models, the equilibrium real rate is typically defined as the quarterly risk-free real rate that would prevail in an economy with fully flexible prices. Furthermore, the distance between the equilibrium real rate and the actual real rate determines the monetary policy stance. As our OLG model is a real model, we implicitly leave out all fluctuations in interest rates due to the business cycle, and the actual real rate coincides with the equilibrium real rate. By extension, we abstract from monetary fluctuations and monetary policy. In fact, the concept of real rates in an OLG framework is closer to the equilibrium return on capital or long-run return to investing in equities than the quarter-on-quarter risk-free interest rate.

Consequently, it is hard, if not non-meaningful, to compare results on the equilibrium real rate in our model to estimates of equilibrium rates from DSGE models or Laubach-Williams type of models.[6] As an example, a study of the $r_t^\star$ for the Danish economy finds a much lower

[5]For an overview of the models and theoretical discussion of equilibrium interest rates, or r*, see Adolfson and Rasmussen (2021).

[6]We could force the model to deliver comparable levels for the real rate within this model by changing the

estimate, see Pedersen (2015). But that model, an open-economy Laubach-Williams type model, also includes movements on business-cycle frequencies as well as long-run changes in productivity. As an example, the model points to an increase in precautionary savings and other shocks driving the business cycle during the aftermath of the great financial crisis as a significant driver of a large fall in $r_t^\star$ during that period, pushing it down towards -2 percent. Clearly, an OLG model as used in the current study cannot capture such drivers of $r_t^\star$, and hence, the results can not be compared in any meaningful way. However, the long-run movements in the equilibrium real rate from our study clearly influence risk-free interest rates that are also of interest for central banks and also influence movements in $r_t^\star$ as estimated in the study mentioned before.

We begin by providing a thorough literature review in Section 1. Next, Section 2 sets up the model that we use to evaluate the effects of the three drivers of the real interest rate. The model is then calibrated using demographic data and projections in Section 3. In Section 4, we show results on the overall change in the real interest rate induced by each and all of the three drivers followed by a discussion of how differential trends in mortality may help explain the build-up of Danish net foreign assets. Finally, Section 6 concludes and discusses avenues for future research.

1. Demographics and Equilibrium Interest Rates: Explaining the Mechanisms

The literature on the implications for interest rates from demographics is already quite developed. Krueger and Ludwig (2007) finds that the equilibrium interest rate will fall internationally from 2005 to 2080 by around 0.90 percentage points. Carvalho et al. (2016), finds a fall by 1.5 percentage points from 1990 to 2014. Comparable effects have been found by Eggertsson et al. (2019) and Gagnon et al. (2021). Papetti (2021) finds that aging can explain a decrease in equilibrium rates of about 1.4 percentage points going toward 2030 compared to the average in the 1980s. Lisack et al. (2017) finds that falling birth and death rates can approximately explain a 1.5 percentage point fall in world real interest rates since the 1980s. Comparable results are found in the study by Bielecki et al. (2020). The role of demographics is confirmed in a panel VAR in Aksoy et al. (2019).

Another potential factor behind the decline in interest rates is changes in the distribution of income and wealth, see e.g. Mian et al. (2020). In a number of countries, the distribution of wealth, in particular, has become more uneven. As the wealthiest part of the population typically has a lower propensity to consume, this has contributed to an increase in global desired savings and hence a decline in equilibrium rates. We observe a pronounced decline in estimates of interest rates in the immediate aftermath of the financial crisis. This is likely to reflect that the crisis led to a general reassessment of risks among firms and households in

subjective discount factor, see also Gagnon et al. (2021), but we have chosen not to and instead focus solely on the long-run movements.

many countries, prompting them to deleverage. With a more cautious approach to savings and spending, real interest rates must fall. However, although intuitively appealing, we do not model the inequality channel in this paper.

When discussing demographics and interest rates, two effects must be distinguished. One is based on shifts in the aggregate age distribution caused by varying sizes of birth cohorts, see e.g.Auerbach and Kotlikoff (1990), Carvalho et al. (2016), Gagnon et al. (2021), and Eggertsson et al. (2019). Within this strand of the literature, demographic changes are mainly driven by the baby boom generation. This birth cohort was particularly large and subsequently had lower fertility than previous birth cohorts. This led to a theoretical exploration of how a large cohort passing through the age distribution affects equilibrium interest rates. One implication from this theory is that as the baby boomers retire and begin to spend their savings, equilibrium interest rates are likely to increase as this cohort is relatively large.

The other effect is the increase in longevity, see Carvalho et al. (2016). Here, the observed rise in life expectancy contributes to increasing savings as individuals prepare for a longer retirement period. All individuals across the age distribution should be expected to save more if everyone expects to have longer retirement periods. The overall effect, of course, depends on adjustments to the retirement age. Through the lens of this theory, equilibrium interest rates do not necessarily increase when the effects of increases in longevity feed through the economy, as the size of the cohorts remains equal in relative terms.

Evaluating the importance of shifts in savings across the age distribution for real interest rates necessitates a model with an overlapping generation structure (OLG model). In such a framework, households in the middle of the age distribution drive the saving behavior of the overall economy. Thus, an increase in the relative size of middle-aged workers pushes up savings and depresses real rates.

To fix ideas, consider for a moment a stylized setting where younger individuals earn nothing, and therefore must borrow from older workers. Further, assume that the older workers consume all of their wealth before dying. In such a setting, borrowing by younger individuals must equal savings by older workers. If the cohort of older workers is large relative to the cohort of younger individuals, interest rates must fall to clear the markets for saving.

One view is that demographic shifts and aging, in particular, affect equilibrium interest rates through their effect on per-capita growth. The logic is that a steady state with lower growth is associated with higher savings today given lower expected income in the future and a desire to smooth consumption. The interest rate must fall to accommodate this larger demand for savings.

In the following, we set up an OLG model for a two-country economy to analyze each of these channels in greater detail.

2. A Two-country OLG Model

The current section presents an overview of the OLG model setup used to disentangle the effects of demography and technological growth on equilibrium interest rates. We note that the theoretical framework does not include business-cycle dynamics, as we focus on long-term movements in demographics and technology. One implication is that the actual real rate coincides with the natural interest rate, and we will consequently not distinguish between actual and natural real rates in what follows.

The model setup includes country-specific home biases, country-specific and dynamic survival probabilities, dynamic fertility growth, dynamic technology growth, a multiperiod lifecycle, a pension system, and accidental bequest. Also, we allow countries to differ in size and technology level. A brief outline of the model goes as follows: There are two countries in the economy; Home and Foreign. The two countries potentially differ in terms of population size, technology level, mortality pattern, and preferences for two country-specific goods. In every calendar year, t, a new cohort of young adults spawns in each country. Individuals in this cohort begin to save and consume in year $t + 1$. Individuals face mortality risk in every period and can potentially live to $t + d$. Individuals work for r years and retire in year $t + r + 1$. While working, the individual earns labor income, pays taxes, receives bequests, consumes, and saves. In retirement, individuals live off of private savings and public pension benefits. The remainder of this section describes each model component in detail.

2.1 Exogenous Processes

We model three exogenous drivers of trends in the world economy and particularly in the real interest rate. First, the technology level in country i is deterministic and follows a dynamic growth process:

$$Z_t^i = Z_0^i \prod_{s=0}^{t-1} \left(1 + g_{z,s}^i\right)$$

where Z_0^i is an initial technology level. We allow technology growth to differ across countries in the short run. In the long run, however, technology rates must converge in order to ensure the existence of a steady state. Fertility is also deterministic, and it too follows a dynamic growth process:

$$n_t^i = n_0^i \prod_{s=0}^{t-1} \left(1 + g_{n,s}^i\right)$$

where n_0^i is the initial population size. As for technology growth, we allow for short-run differences in cohort-size growth but impose convergence in the long run. In terms of modeling mortality, we denote the unconditional survival rate for individuals born at time t

to survive to age τ by:

$$\Psi^i_{\tau,t} = \prod_{s=1}^{\tau} \psi^i_{s,t}$$

Here, $\psi^i_{s,t}$ is the likelihood of living to age s conditional on having already lived to age $s-1$. Importantly, we do not track individual deaths but rather assume that each generation is large enough for us to invoke the law of large numbers. Under this assumption, the actual number of deaths is equal to the expected number of deaths for every generation at every point in time. Thus, each country consists of $N^i_t = \sum_{\tau=1}^{d} \Psi^i_{\tau,t-\tau} n^i_{t-\tau}$ living individuals. Out of the living population, $L^i_t = \sum_{\tau=1}^{r} \Psi^i_{\tau,t-\tau} n^i_{t-\tau}$ individuals are in the workforce, while $O^i_t = \sum_{\tau=r+1}^{d} \Psi^i_{\tau,t-\tau} n^i_{t-\tau}$ are retired.

2.2 The Individual Utility Maximization Problem

Individuals derive utility from a composite good that consists of two country-specific goods. Therefore, the utility maximization problem can be broken into two separate stages; i) an intertemporal stage and ii) an intratemporal stage. In the intertemporal stage, the individual chooses the optimal allocation of composite goods over the life cycle. In the intratemporal stage, the individual maximizes the amount of composite goods over input quantities of country-specific goods within the current period.

2.2.1 Intertemporal Problem

An individual born in country i at time t makes consumption-saving decisions to maximize a stream of discounted and probability-weighted instantaneous utilities from consuming a composite good, $C^i_{\tau,t}$, over the life cycle. Thus, individuals also choose savings measured in composite goods.[7] Consequently, we denote factor prices in terms of country-composite goods. Formally, the intertemporal problem reads:

$$\max_{\left\{C^i_{\tau,t}, S^i_{\tau,t}\right\}} U^i_t = \sum_{\tau=1}^{d} \beta^\tau \Psi^i_{\tau,t} u\left(C^i_{\tau,t}\right)$$

$$s.t.$$

$$C^i_{\tau,t} + S^i_{\tau,t} = W^i_{t+\tau} l^i_{\tau,t} + R^i_{t+\tau} S^i_{\tau-1,t} + B^i_{t+\tau}$$

While working, the individual earns labor income, $W^i_{t+\tau}$, denoted in country-specific composite goods. Here, $l^i_{\tau,t}$ is an indicator variable that allows for an exogenous retirement age. Individuals carry savings, $S^i_{\tau-1,t}$, into the current period at the going interest rate, $R^i_{t+\tau}$. Finally, all living individuals receive an age-independent bequest, $B^i_{t+\tau}$, in every period. For more on the bequest mechanism, see Section 2.3. Out of all this income, the individual saves

[7]Later, we introduce a transformation of savings into productive capital based on a common price.

$S^i_{\tau,t}$ and thereby consumes $C^i_{\tau,t}$. The solution to the intertemporal problem is standard with the short-term Euler equation determining the optimal consumption path. Assuming CES utility, the corresponding long-term Euler equation reads:

$$C^i_{\tau,t} = C^i_{1,t}\left(\beta^{\tau-1}\Psi^i_{\tau,t}\frac{\prod_{s=1}^{\tau} R^i_{t+s}}{R^i_{t+1}}\right)^{\frac{1}{\sigma}}$$

The equilibrium consumption path is pinned down by the consolidated budget:

$$\sum_{\tau=1}^{d}\frac{C^i_{\tau,t}}{\prod_{s=1}^{\tau} R^i_{t+s}} = \sum_{\tau=1}^{d}\frac{W^i_{t+\tau}l^i_{\tau,t} + B^i_{t+\tau}}{\prod_{s=1}^{\tau} R^i_{t+s}}$$

Combining the long-term Euler equation and the consolidated budget, we compute the closed-form solution for first-period consumption for given prices:

$$C^i_{1,t} = \frac{\sum_{\tau=1}^{d}\frac{W^i_{t+\tau}l^i_{\tau,t} + B^i_{t+\tau}}{\prod_{s=1}^{\tau} R^i_{t+s}}}{\sum_{\tau=1}^{d}\frac{\left(\frac{\beta^{\tau-1}\Psi^i_{\tau,t}\prod_{s=1}^{\tau} R^i_{t+s}}{R^i_{t+1}}\right)^{\frac{1}{\sigma}}}{\prod_{s=1}^{\tau} R^i_{t+s}}}$$

The full path of consumption and saving then follows from iterating the Euler equation and the period-by-period budget forward in time.

2.2.2 Intratemporal Problem

In the intratemporal problem, individuals choose the optimal combination of country-specific goods, $\left(C^{i,i}_{\tau,t}, C^{i,j}_{\tau,t}\right)$, used to construct a composite good, $C^i_{\tau,t} = C\left(C^{i,i}_{\tau,t}, C^{i,j}_{\tau,t}\right)$. Following the bulk of the macro-trade literature, composite goods are modeled as the outputs of an Armington-style aggregator:

$$C\left(C^{i,i}_{\tau,t}, C^{i,j}_{\tau,t}\right) = \left(\gamma_i^{\frac{1}{\eta}}\left(C^{i,i}_{\tau,t}\right)^{\frac{\eta-1}{\eta}} + (1-\gamma_i)^{\frac{1}{\eta}}\left(C^{i,j}_{\tau,t}\right)^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}$$

where γ_i controls the degree of home bias in country i, and η measures the degree of substitution between the two country-specific goods in the production of composite goods. Now, suppose that an individual living in Home and an individual living in Foreign both have an income worth exactly one Home-specific good. Denoting by p_t the relative price of Foreign goods, this is equivalent to an income of $\frac{1}{p_t}$ Foreign goods. For the individual living in Home, one can show that the optimal allocation of Home-specific income into

Home-specific and Foreign-specific goods in producing Home-composite goods is given by:

$$
\begin{aligned}
x_t^{h,h} &= \frac{1}{1 + p_t^{1-\eta} \dfrac{1-\gamma_h}{\gamma_h}} \\
x_t^{h,f} &= \frac{1}{p_t + p_t^{\eta} \dfrac{\gamma_h}{1-\gamma_h}}
\end{aligned}
$$

For future use, we refer to these as the optimal input quantities in producing Home-composite goods. Conversely, the optimal input quantities in the production of Foreign-composite goods are given by:

$$
\begin{aligned}
x_t^{f,f} &= \frac{1}{p_t + p_t^{\eta} \dfrac{1-\gamma_f}{\gamma_f}} \\
x_t^{f,h} &= \frac{1}{1 + p_t^{1-\eta} \dfrac{\gamma_f}{1-\gamma_f}}
\end{aligned}
$$

Optimal input quantities behave very intuitively. If the good produced in a country becomes relatively more expensive for some reason, people in both countries demand less of it for each unit of income. The effect on the other good depends on the elasticity of substitution in the aggregator for both countries. This is the standard income versus substitution effect mechanism. Plugging the optimal input quantities back into the aggregator, one can show that an income of one Home-specific good can be transformed into the following amounts of country-composite goods:

$$
\begin{aligned}
\pi_t^h &= \left[\gamma_h \left(1 + p_t^{1-\eta} \frac{1-\gamma_h}{\gamma_h}\right)\right]^{\frac{1}{\eta-1}} \\
\pi_t^f &= \left[\left(1-\gamma_f\right)\left(1 + p_t^{1-\eta} \frac{\gamma_f}{1-\gamma_f}\right)\right]^{\frac{1}{\eta-1}}
\end{aligned}
$$

Throughout the remainder of the paper, these inverse prices of composite goods in terms of country-specific goods serve as a useful means of converting country-specific prices into country-composite prices and vice versa. For instance, it is useful to remember that an income of one composite good corresponds to an income of $1/\pi_t^i$ Home-specific goods and uses $x_t^{i,h}$ Home-specific goods and $x_t^{i,f}$ Foreign-specific goods in its production.

2.3 Bequest

For completeness, unclaimed savings of the dead are redistributed among the living as accidental bequest. As the model incorporates no formal dynastic links between generations, we assume that accidental bequest is evenly distributed among all the living. Thus, the amount of bequest received is age-independent and reads:

$$B_{t+1}^{i} = \frac{R_{t+1}^{i} \sum_{\tau=1}^{d} \left(\Psi_{\tau,t-\tau}^{i} - \Psi_{\tau+1,t-\tau}^{i} \right) n_{t-\tau}^{i} S_{\tau,t-\tau}^{i}}{\sum_{\tau=1}^{d} \Psi_{\tau,t+1-\tau}^{i} n_{t+1-\tau}^{i}}$$

Such an assumption may be somewhat unrealistic but ensures the preservation of resources.

2.4 International Capital Market Clearing

As always, savings are used to produce capital. In this model, however, savings are denominated in two different composite goods that may change in composition over time. To avoid valuation or terms-of-trade effects in capital, we assume that productive capital is based on a common technology. Particularly, we transform savings in composite goods into productive capital at a rate equal to the value of savings measured in Home-specific goods. Formally, as one Home-specific good buys π_t^i country-composite goods, capital owned by country i reads:

$$K_{t+1}^{i} = \frac{S_t^i}{\pi_t^i}$$

where $S_t^i = \sum_{\tau=1}^{d} \Psi_{\tau,t-\tau}^{i} n_{t-\tau}^{i} S_{\tau,t-\tau}^{i}$ denotes total composite-good savings in country i. As capital is completely mobile, capital owned by one country needs not equal the amount of capital allocated to that country. Nonetheless, international capital markets must clear. Denoting by $\mathcal{K}_t^i$ all productive capital allocated to country i, international capital-market clearing requires:

$$\mathcal{K}_{t+1}^{h} + \mathcal{K}_{t+1}^{f} = K_{t+1}^{h} + K_{t+1}^{f}$$

Naturally, this implies that the net-foreign asset positions of the two countries cancel out.

2.5 Firms

Production is iso-elastic and depends on capital and effective units of labor. To ensure the existence of a steady-state growth path, technology is assumed to be Harrod-neutral (labor-augmenting). The production function reads:

$$Y_t^i = A \left(\alpha \left(\mathcal{K}_t^i \right)^{\frac{\theta-1}{\theta}} + (1-\alpha) \left(Z_t^i L_t^i \right)^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}$$

Factor prices come from the profit-maximization problem of firms in the two countries. Firm profit reads:

$$\Pi_t^h = Y_t^h - \rho_t^h \mathcal{K}_t^h - \omega_t^h L_t^h$$
$$\Pi_t^f = p_t Y_t^f - \rho_t^f \mathcal{K}_t^f - \omega_t^f L_t^f$$

where ρ_t^i and ω_t^i are the capital and labor rental rates in Home-specific equivalents for country i. The capital rental rate is composed of a country-specific real interest rate, r_t^i, and a worldwide depreciation rate, δ. By profit maximization, the country-specific capital rental rates denominated in Home-specific goods can be written as:

$$r_t^h + \delta = \alpha A^{\frac{\theta-1}{\theta}} \left(\frac{Y_t^h}{\mathcal{K}_t^h} \right)^{\frac{1}{\theta}}$$
$$r_t^f + \delta = p_t \alpha A^{\frac{\theta-1}{\theta}} \left(\frac{Y_t^f}{\mathcal{K}_t^f} \right)^{\frac{1}{\theta}}$$

As savings are completely mobile, however, the real returns on savings must equalize at r_t. Factor-price equalization requires the following relative price:

$$p_t = \left(\frac{Y_t^h \mathcal{K}_t^f}{Y_t^f \mathcal{K}_t^h} \right)^{\frac{1}{\theta}}$$

As it is composite factor prices that figure into the individual maximization problem, we need to express prices in composite terms. For a given world interest rate, r_t, denominated in Home-specific goods, the country-composite interest rate reads:

$$R_t^i = (1 + r_t) \frac{\pi_t^i}{\pi_{t-1}^i}$$

Intuitively, each composite good saved buys $\frac{1}{\pi_{t-1}}$ Home-specific goods worth of capital. In the next period, this gives a gross return of $(1 + r_t)\, \pi_t^i$ composite goods. For wages in per unit of labor terms, profit maximization again implies:

$$\omega_t^h = Z_t^h (1 - \alpha) A^{\frac{\theta-1}{\theta}} \left(\frac{Y_t^h}{Z_t^h L_t^h} \right)^{\frac{1}{\theta}}$$
$$\omega_t^f = Z_t^f p_t (1 - \alpha) A^{\frac{\theta-1}{\theta}} \left(\frac{Y_t^f}{Z_t^f L_t^f} \right)^{\frac{1}{\theta}}$$

This translates into the following wages per unit of labor in country-composite terms:

$$W_t^h = \pi_t^h \omega_t^h$$
$$W_t^f = \pi_t^f \omega_t^f$$

2.6 Goods Market Clearing

Both country-specific goods available from either current production or leftover capital must either be consumed or saved to produce new capital. Goods-market clearing requires:

$$Y_t^i + (1-\delta) \sum_{j \in h,f} \frac{x_{t-1}^{j,i}}{\pi_{t-1}^j} S_{t-1}^j = \sum_{j \in h,f} \frac{x_t^{j,i}}{\pi_t^j} \left(C_t^j + S_t^j \right)$$

where $C_t^i = \sum_{\tau=1}^{d} \Psi_{\tau,t-\tau}^i n_{t-\tau}^i C_{\tau,t-\tau}^i$ is aggregate composite good consumption across all generations within a country.

2.7 Solving the Model

In this section, we describe how to compute the dynamic transition path from one steady state to another in response to an MIT shock to technology and demographic variables.

To initialize the experiment and our solution algorithm, we assume that the economy is initially in a growth-adjusted steady state and that it has been so for many years prior. For such an equilibrium to exist, we specifically assume that everyone born before 1900 was born with similar demographic characteristics. As these people belong to different cohorts, we use 1950 period mortality rates as a proxy for 1900 cohort rates. At this point in time, the secular drivers: mortality, fertility, and technological growth, suddenly begin to change. The secular change lasts for 100 years and thus ends in the year 2000. To be consistent, cohorts born after this period are all born with the projected period mortality rates of 2050. The change in the secular drivers leads to a demographic transition, which ends only $d = 80$ periods later when the demographic profile again becomes constant. After the demographic transition ends, economic variables continue to adjust toward the long-run equilibrium. To compute the dynamic transition path, we truncate an infinitely forward-looking problem by imposing that the economy converges to a new steady state over some finite horizon. To avoid restricting the transition path, truncation occurs in the year 2250. To circumvent steady-state end-point problems obfuscating our results, we focus on the part of the dynamic transition path between 1950 and 2050.

To compute the growth-adjusted steady states, we define the growth-adjusted version of all aggregate variables as $x_{t+\tau} = \frac{X_{t+\tau}}{Z_{t+\tau} L_{t+\tau}}$. Moreover, we define the growth-adjusted version of individual variables as $x_{\tau,t} = \frac{X_{\tau,t}}{Z_{t+\tau}}$. For the wage specifically, $w_{t+\tau}^i = \frac{W_{t+\tau}^i}{Z_{t+\tau}}$. The growth-corrected equations are described in detail in Appendix A. This allows us to write a set of equations that define the equilibrium anywhere on the adjustment path. We then use the growth-corrected model equations to solve for the steady state. This is defined as a situation where the mortality pattern, population growth, and technology growth have all become constant, and growth-adjusted variables have subsequently converged. The steady-state solution is described in detail in Appendix B.

To solve for the dynamic transition path under perfect foresight, we then apply a relaxation algorithm combined with a standard updating mechanism. First, we produce an initial guess for the dynamic transition path. We then continue to update our guess under optimal behavior and market-clearing conditions until all aggregate variables converge. For consistency, we introduce an updating mechanism that allows individuals that are alive when the MIT shock arrives to revise the rest of their original consumption plan. This is done in Appendix B.1.

3. Calibration

This section briefly summarizes the general calibration of parameters and two different experiments for the dynamic transition path. We begin in a setting with symmetric shocks to demographic variables in both countries to inspect the main global drivers of real interest rates and to compare these to the results of the existing literature. We consider three driving forces: a fall in the growth rate of technology, a fall in the growth rate of the population, and an increase in survival. Subsequently, we consider asymmetric shocks to mortality rates to quantify the effect on the net foreign asset position of the two countries.

US mortality data are taken from the Social Security Administration. The survival rates are shown in Figure 1. Danish mortality data is provided by DREAM.[8] To weed out the effects of wars and pandemics that may temporarily affect longevity, we consider a simple interpolation of age-specific survival rates from 1950 levels to 2050 levels. For population growth, we adjust the fertility rate to match old-age dependency ratios reported by the OECD in both the initial and the long-run steady state. Figure 2 depicts old-age dependency ratios for the US and Denmark in the calibrated model as compared to the data. This gives us a fertility rate equal to 1 percent in 1950. In the long run, this cohort growth rate declines to -0.2 percent. On the transition path, we again connect the two steady states by interpolation. Finally, we assume that technology growth in 1950 was 2 percent per annum and that technology growth declines to 1 percent per annum towards 2050. This is in line with the existing literature. Again, the two equilibria are connected by interpolation. Denmark is assumed to follow a similar pattern.

We calibrate the size of the US economy relative to the Danish economy to a factor of fifty to one. The home-bias parameters are also calibrated according to the relative size of the two economies. Accordingly, Denmark has a small bias, whereas the US has a large bias. Calibration to relative sizes has the advantage that countries with similar parameters

[8] We are using data compiled at the beginning of 2022.

Parameter	Description	Value	Origin
Preferences			
β	Discount Factor	0.99	Literature
σ	EIS of Consumption	1.5	Literature
γ_h	Home Bias in Home	50/51	Relative Size
γ_f	Home Bias in Foreign	1/51	Relative Size
η	Elasticity of Substitution	1.5	Literature
Production and Capital			
α	Capital Share	0.25	Literature
θ	EIS Parameter	1.5	Literature
δ	Capital Depreciation Rate	0.07	Literature
Exogenous Processes			
n_0^h	Initial Population Size US	50	Relative Size US
n_0^f	Initial Population Size DK	1	Relative Size DK
z_0^h	Initial Technology US	1	Similar Tech.
z_0^f	Initial Technology DK	1	Similar Tech.
$g_{n,1900}$	1900 Population Growth	1 percent	Matching OADP
$g_{n,2000}$	2000 Population Growth	-0.2 percent	Matching OADP
$g_{n,1900-2000}$	Dynamic Transition	Interpolation	Interpolation
$g_{z,1900}$	1900 Technology Growth	2 percent	Historical
$g_{z,2000}$	2000 Technology Growth	1 percent	Exploratory
$g_{z,1900-2000}$	Dynamic Transition	Interpolation	Interpolation
$\Psi^h_{\tau,1900}$	1900 Cohort Mortality	Period Mortality 1950	SSA
$\Psi^h_{\tau,2000}$	2000 Cohort Mortality	Period Mortality 2050	SSA Projection
$\Psi^h_{\tau,1900-2000}$	Dynamic Transition	Interpolation	Interpolation
$\Psi^f_{\tau,1900}$	1900 Cohort Mortality DK	US Data	Exposition
$\Psi^f_{\tau,2000}$	2000 Cohort Mortality DK	DREAM	Projection
$\Psi^f_{\tau,1900-2000}$	Dynamic Transition	Interpolation	Interpolation

Table 1: Calibration of parameters in the model.

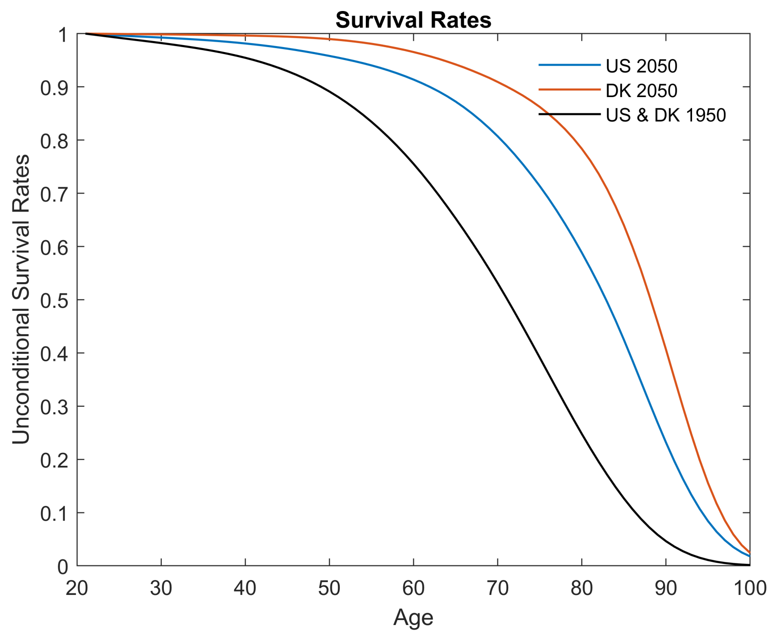

Figure 1: **Survival Rates for the US and Denmark**
The figure shows the unconditional survival rates at all ages used in the simulations. For Denmark, data is provided by DREAM. For the US, the data is from the Social Security Administration.

end up in an equilibrium with a relative price of one and zero net foreign asset positions. Only for asymmetric calibrations, prices and net foreign asset positions diverge from this equilibrium. All remaining parameters pertaining to production and consumption are calibrated to standard values. See Table 1.

4. Symmetric Shocks to the Global Economy

In this section, we show results computed under the assumptions of symmetric shocks to the three drivers on a global level. That is, for now, the two countries are identical and face the exact same shocks. To reiterate, the shocks that we consider are an increase in longevity, a decrease in fertility, and a slowdown in technology growth. In this symmetric case, mortality rates in both countries are assumed to follow the US pattern as shown in Figure 1.

For the given calibration, the model produces quite realistic outcomes. To see this, consider Figure 3. For instance the capital-output ratio increases from 3.14 in 1950 to around 4.40 in 2050. In response, the real interest rate decreases from around 4.7 to around 2.3. Hence, falling population growth, mortality rates, and technology growth depress the equilibrium real interest rate by 2.25 percentage points from 1950 to the present. Thus, according to the model, we should expect a further fall in the equilibrium real rate by 0.25 in the future based alone on the rolling-over of cohorts that goes on after the initial transition period. If we were to account for further improvements in period mortality rates beyond

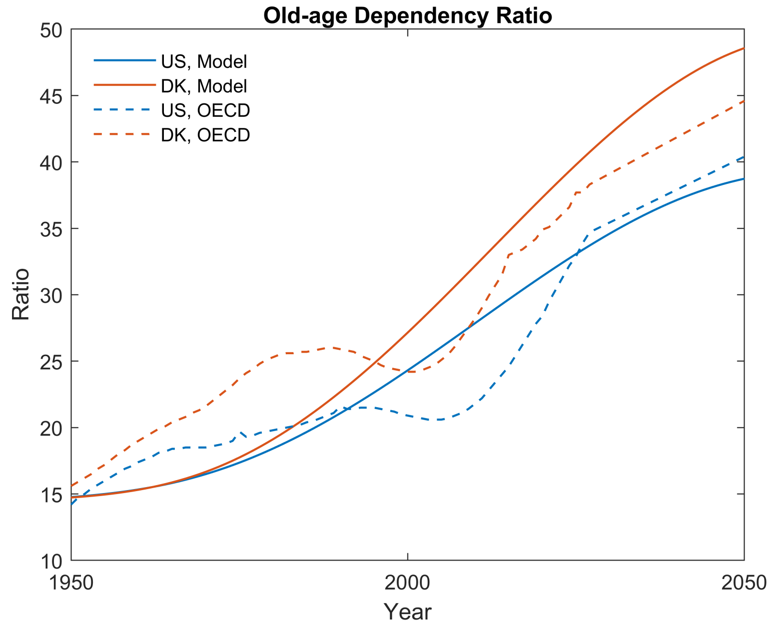

Figure 2: **Old-age Dependency Ratios**
The figure shows Old-age Dependency Ratios for the US (blue) and Denmark (red) in the calibrated model (solid) and according to the OECD (dashed).

2050, this subsequent decline would only be more pronounced. In other words, although dynamics on the business cycle frequency can affect the equilibrium real interest rate in both directions in the short run, the overall tendency in equilibrium rates is a further decline.

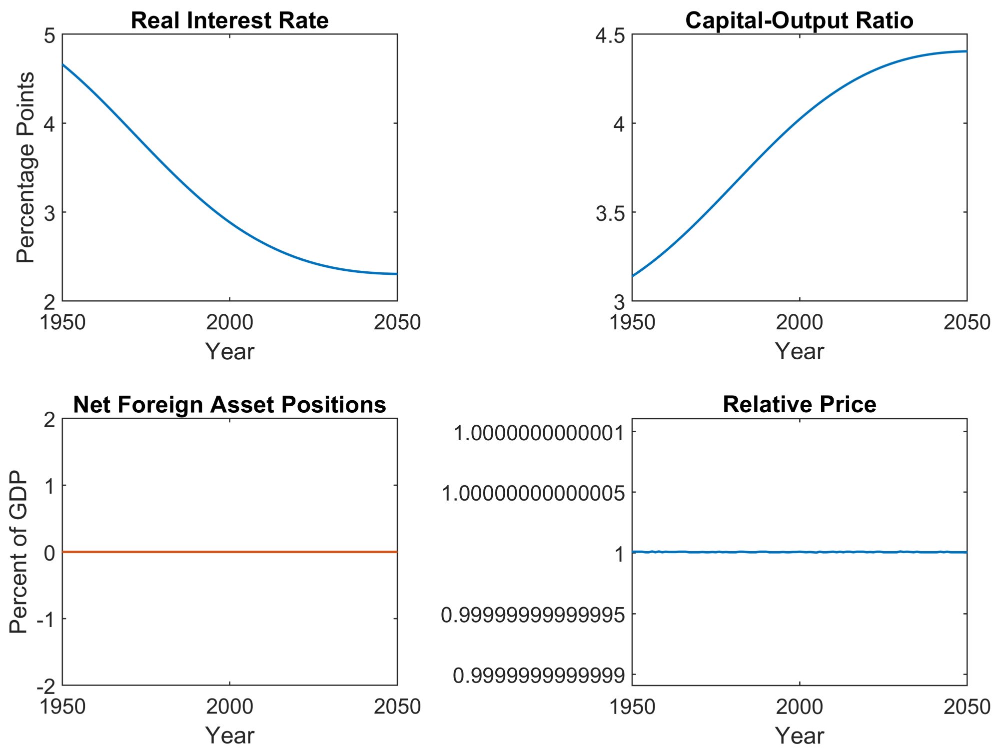

Figure 3: **Reaction in Other Variables**

This finding is important both from a market and policy perspective. Currently, inflation has reached around 10 percent in many developed countries, and policy rates are being raised rapidly, pushing up the yield curve. But our results point to a return to a low-interest environment when inflationary shocks have abated. In other words, the current high inflation environment is likely to be seen as an anomaly looking backward 50 years from now, naturally under the assumption that central banks will manage to get inflation under control during the current business cycle. Our results indicate that when inflation eventually falls back to its target and the output gap closes, the policy rate needs to fall even further than before to close the output gap and bring inflation back on target when a hypothetical contractionary demand shock hits the economy.

As the countries are modeled symmetrically with home bias proportional to relative country sizes, relative prices, and real exchange rates are fixed at unity, and net foreign asset positions are zero. In Figure 4, we isolate the impact of each individual driver by including all drivers except for that of interest and comparing this to the total effect. All the drivers affect the real interest rate negatively. The main driver of the fall in the real interest rate is the fall in mortality. Disregarding improvements in longevity and looking only at the impact of the slowdown in fertility and technology growth reduces the fall in the real interest rate by 2050 by around 1.6 points (change in the difference between the blue and purple line in 2050). In a similar fashion, ruling out changes in fertility, the fall in the real interest rate decreases

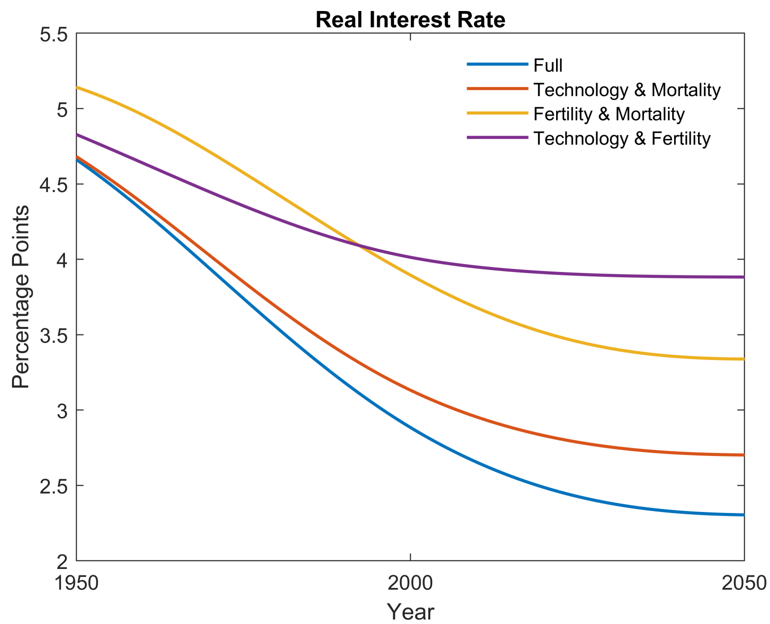

Figure 4: **Real Interest Rate Response to Changes in Mortality, Fertility, and Technology Growth**
The blue line shows the total effect. The isolated effects of each driver are then retrieved by shutting off one driver at a time. The difference between each line and the total effect measures the impact of the driver in question. First, the red line shuts off the fertility channel. Second, the yellow line shuts off the technology channel. Finally, the purple line shuts off the mortality channel.

by 0.4 percentage points. Meanwhile, abstracting from the slowdown of technology growth reduces the reaction in the real interest rate by 1.0 percentage points. Of course, these results rely on our assumptions regarding the different growth rates. That is, the changes in the driving forces are not all comparable in size. Also, the analysis may depend on the reference values of the different drivers. The figure also illustrates that the effect of mortality kicks in later than the changes in fertility and technology, as older generations are replaced by younger ones even after the mortality rates of new cohorts have become constant.

5. Danish Autarky Interest Rates and Asymmetric Shocks

In the previous analysis, we looked at the global equilibrium real rate, taking the US as a proxy for the world economy. Moving on, we devote this section to analyzing what the equilibrium rate in Denmark would look like if we shut off the impact of the global demographic transition, e.g., the Danish autarky equilibrium interest rate. Figure 5 depicts the results of this exercise. According to the model, the Danish autarky equilibrium interest rate is around 0.6 percentage points below the global equilibrium rate. This difference is due to a stronger increase in longevity in Denmark, pushing up savings by more than in the case of the global economy. As shown in the second panel, this goes hand-in-hand with a higher capital-output ratio.

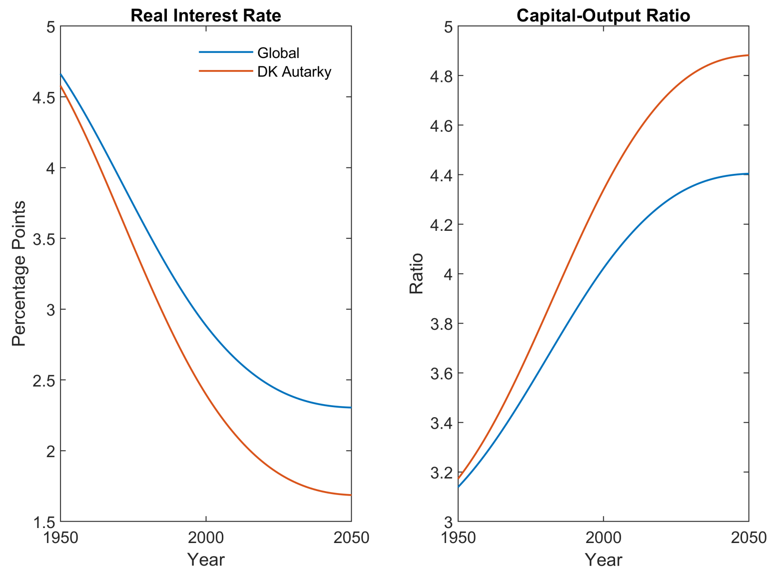

Figure 5: **Autarky Interest Rates**
The blue line depicts the case of the global economy following US projections. The red line shows the response of the Danish equilibrium real rate and capital-output ratio, assuming that Denmark is a closed economy, that productivity falls, and that the demographic transition happens as shown in Figure 1.

Finally, we provide insight into how differences in global and local developments affect the Danish economy. Here, we pay particular attention to the build-up of the Danish net foreign asset position due to increasing current account surpluses over the past two to three decades. In 2022, the current account surplus will likely reach around 10 percent of GDP.

As in the previous exercise, we let the growth rate in technology and population fall symmetrically in the two countries. However, now we impose US mortality predictions in the large country and Danish mortality predictions in the small country, as shown in Figure 1. This asymmetry induces imbalances in capital ownership across borders. Results are depicted in Figure 6. The interest rate differs little from the symmetric case, as Denmark only constitutes a small fraction of the world economy. Interestingly, the Danish net foreign asset position goes from 0 (by construction) in 1950 to around 135 percent of GDP in the long-run equilibrium. In 2021, the model predicts Danish net foreign assets to be roughly 100 percent of GDP. For comparison, Denmark's actual net foreign asset position against the rest of the world was around 76 percent of GDP in 2021. There are several potential reasons for the model to overshoot on this measure, i.e., inaccurate projections and calibrations, real-world restrictions of international capital flows, real-world periods of financial imbalances, etc. Nonetheless, the exercise shows that differences in longevity can serve as a strong driver of the build-up of Danish net foreign assets vis-à-vis the global economy in the long run. On the other hand, the net foreign asset position relative to GDP in the rest of the world moves relatively little, as the rest of the world makes up most of the world economy. In addition,

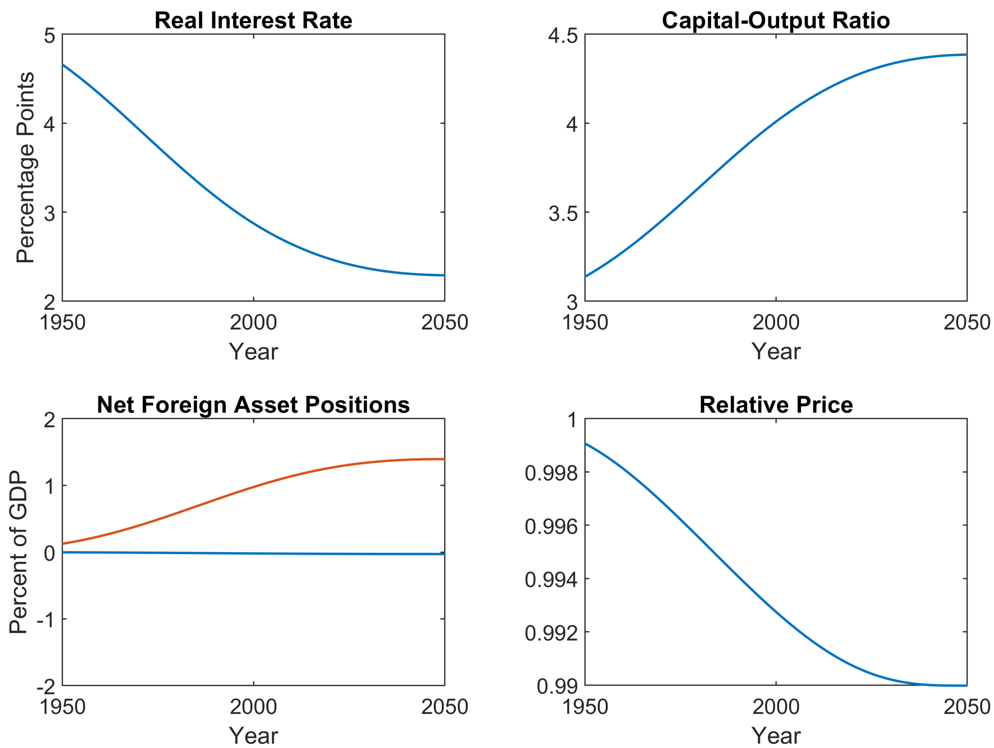

Figure 6: **Reaction in Other Variables**

the asymmetric shocks induce a terms-of-trade effect as the relative price of foreign goods decreases from unity to around 0.99. Ceteris paribus, this terms-of-trade effect makes it cheaper to buy Danish goods and improves competitiveness. However, it is difficult to say whether the data reflects this effect, as real-world competitiveness depends on factors other than demographics and growth.

That the increase in longevity in Denmark is a strong driver of the Danish net foreign asset position is an interesting result, as it implies that the accumulation of assets reflects a desire for the country to save for future retirement. However, using the simple model, it is important to stress that we cannot rule out that other driving forces play and have played a role in the savings behavior of Danish households.

6. Conclusion and Discussion

We build a model for the global economy separated into two regions. Subsequently, we calibrate the two regions into a large open economy and a small open economy. In a first experiment, we calibrate the entire world economy according to US data and examine the effect of three drivers - a slowdown of population growth, a slowdown of growth in technology, and increased longevity - on the global real interest rate. We find that these three drivers lead to a decrease in the real interest rate of around 2.4 percentage points going from 1950 to 2050. Further, while there is evidence in favor of a historical fall in equilibrium real rates, our results suggest that this fall has not ended. Specifically, based solely on the rolling-over of generations after the transition in cohort longevity and fertility, the real interest rate will likely fall by a further 0.25 percentage points. If we were to model improvements in period survival rates beyond 2050, the fall would be even larger. In a second experiment, we calibrate the small open economy on Danish data and show that differences in longevity may help to explain the build-up of Danish net foreign assets vis-à-vis the US.

There are several avenues for future research. One such avenue relates to the assumption of interest rate parity and the free movement of capital. With unrestricted capital flows between the two economies and factor-price equalization, the real interest rate is primarily determined by developments in the larger of the two economies. While interest rates and general financial conditions are most affected by global trends, domestic shocks can also play a role.[9] There are various approaches for breaking the interest rate parity. One strand of the literature restricts quantities through borrowing constraints. See, e.g., Eggertsson et al. (2019). Under such a formulation, households and firms simply are not allowed to borrow beyond a certain debt level. Another strand of the literature imposes exogenous risk premia, which increases with the indebtedness of the economy. This approach is equally ad-hoc as there is no default risk and thus no need for risk premia when investing in international bonds. For simplicity, we abstract from risk premia in interest rates and focus on the world interest rate and capital ownership.

[9]See Adolfsen and Pedersen (2019), where movements in Danish $r_t^\star$ are shown to be closely correlated with movements in global $r_t^\star$ defined as an average of $r_t^\star$ in the Euro Area and in the US. But at the same time, large and persistent discrepancies in $r_t^\star$ across countries happen frequently.

References

Jakob Feveile Adolfsen and Jesper Pedersen. The natural real interest rate in Denmark has declined. Technical report, Danmarks Nationalbank Analysis no. 13, 27th of June, 2019.

Claus Brand, Marcin Bielecki, and Adrian Penalver. The natural rate of interest: estimates, drivers, and challenges to monetary policy. *ECB Occasional Paper*, (217), 2018.

Coen Teulings and Richard Baldwin. Secular stagnation: Facts, causes, and cures–a new vox ebook. *London: United Kingdom: Centre for Economic Policy Research*, 2014.

Marcin Bielecki, Michał Brzoza-Brzezina, and Marcin Kolasa. Demographics and the natural interest rate in the euro area. *European Economic Review*, 129:103535, 2020.

Noëmie Lisack, Rana Sajedi, and Gregory Thwaites. Demographic trends and the real interest rate. 2017.

Pedersen Jesper Adolfson, Jakob and Thea Thalia Iuel Rasmussen. How does Covid-19 affect r*? Technical report, Danmarks Nationalbank Economic Memo, 2021.

Etienne Gagnon, Benjamin K Johannsen, and David López-Salido. Understanding the new normal: the role of demographics. *IMF Economic Review*, 69(2):357–390, 2021.

Jesper Pedersen. The Danish natural real rate of interest and secular stagnation. Technical report, Danmarks Nationalbank Working Papers, 2015.

Dirk Krueger and Alexander Ludwig. On the consequences of demographic change for rates of returns to capital, and the distribution of wealth and welfare. *Journal of Monetary Economics*, 54(1):49–87, 2007.

Carlos Carvalho, Andrea Ferrero, and Fernanda Nechio. Demographics and real interest rates: Inspecting the mechanism. *European Economic Review*, 88:208–226, 2016.

Gauti B Eggertsson, Neil R Mehrotra, and Jacob A Robbins. A model of secular stagnation: Theory and quantitative evaluation. *American Economic Journal: Macroeconomics*, 11(1): 1–48, 2019.

Andrea Papetti. Demographics and the natural real interest rate: historical and projected paths for the euro area. *Journal of Economic Dynamics and Control*, page 104209, 2021.

Yunus Aksoy, Henrique S Basso, Ron P Smith, and Tobias Grasl. Demographic structure and macroeconomic trends. *American Economic Journal: Macroeconomics*, 11(1):193–222, 2019.

Atif R Mian, Ludwig Straub, and Amir Sufi. Indebted demand. Technical report, National Bureau of Economic Research, 2020.

Alan J Auerbach and Laurence J Kotlikoff. Demographics, fiscal policy, and US saving in the 1980s and beyond. *Tax policy and the economy*, 4:73–101, 1990.

A. Dynamic Solution Strategy

The dynamic solution strategy then goes as follows. In any period, we compute aggregate savings based on individual savings. Aggregate savings are then transformed into next-period capital via the common transformation technology, while leftover wealth from the departed is redistributed as accidental bequest:

$$s_t^i = \frac{\sum_{\tau=1}^d \Psi^i_{\tau,t-\tau} n^i_{t-\tau} s^i_{\tau,t-\tau}}{L_t^i} \tag{1}$$

$$k_{t+1}^i = \frac{s_t^i}{\pi_t^i} \frac{Z_t^i L_t^i}{Z_{t+1}^i L_{t+1}^i} \tag{2}$$

$$b_{t+1}^i = \frac{Z_t^i}{Z_{t+1}^i} \frac{R_{t+1}^i \sum_{\tau=1}^d \left(\Psi^i_{\tau,t-\tau} - \Psi^i_{\tau+1,t-\tau}\right) n^i_{t-\tau} s^i_{\tau,t-\tau}}{\sum_{\tau=1}^d \Psi^i_{\tau,t+1-\tau} n^i_{t+1-\tau}} \tag{3}$$

Suppose then that we guess an allocation of capital $\left(\kappa^h_{t+1}, \kappa^f_{t+1}\right)$ that is consistent with the adjusted capital market-clearing condition:

$$0 = \left(\kappa^h_{t+1} - k^h_{t+1}\right) + \left(\kappa^f_{t+1} - k^f_{t+1}\right) \frac{Z^f_{t+1} L^f_{t+1}}{Z^h_{t+1} L^h_{t+1}} \tag{4}$$

In that case, we can compute all of the following aggregate variables sequentially:

$$y_{t+1}^{i} = A\left(\alpha\left(\kappa_{t+1}^{i}\right)^{\frac{\theta-1}{\theta}} + (1-\alpha)\right)^{\frac{\theta}{\theta-1}} \tag{5}$$

$$p_{t+1} = \left(\frac{y_{t+1}^{h}\kappa_{t+1}^{f}}{y_{t+1}^{f}\kappa_{t+1}^{h}}\right)^{\frac{1}{\theta}} \tag{6}$$

$$\pi_{t+1}^{h} = \left[\gamma_h\left(1 + p_{t+1}^{1-\eta}\frac{1-\gamma_h}{\gamma_h}\right)\right]^{\frac{1}{\eta-1}} \tag{7}$$

$$\pi_{t+1}^{f} = \left[\left(1-\gamma_f\right)\left(1 + p_{t+1}^{1-\eta}\frac{\gamma_f}{1-\gamma_f}\right)\right]^{\frac{1}{\eta-1}} \tag{8}$$

$$x_{t+1}^{h,h} = \frac{1}{1 + p_{t+1}^{1-\eta}\frac{1-\gamma_h}{\gamma_h}} \tag{9}$$

$$x_{t+1}^{h,f} = \frac{1}{p_{t+1} + p_{t+1}^{\eta}\frac{\gamma_h}{1-\gamma_h}} \tag{10}$$

$$x_{t+1}^{f,f} = \frac{1}{p_{t+1} + p_{t+1}^{\eta}\frac{1-\gamma_f}{\gamma_f}} \tag{11}$$

$$x_{t+1}^{f,h} = \frac{1}{1 + p_{t+1}^{1-\eta}\frac{\gamma_f}{1-\gamma_f}} \tag{12}$$

$$r_{t+1} = \alpha A^{\frac{\theta-1}{\theta}}\left(\frac{y_{t+1}^{h}}{\kappa_{t+1}^{h}}\right)^{\frac{1}{\theta}} - \delta \tag{13}$$

$$R_{t+1}^{i} = \left(1 + r_{t+1}\right)\frac{\pi_{t+1}^{i}}{\pi_{t}^{i}} \tag{14}$$

$$w_{t+1}^{h} = \pi_{t+1}^{h}\left(1-\alpha\right)A^{\frac{\theta-1}{\theta}}\left(y_{t+1}^{h}\right)^{\frac{1}{\theta}} \tag{15}$$

$$w_{t+1}^{f} = \pi_{t+1}^{f}p_{t+1}\left(1-\alpha\right)A^{\frac{\theta-1}{\theta}}\left(y_{t+1}^{f}\right)^{\frac{1}{\theta}} \tag{16}$$

Atomistic individuals have perfect foresight and expect the path of future factor prices, $\left(r_{t+j}, w_{t+j}\right)_{j=1}^{d}$, and the path of future bequests, $\left(b_{t+j}^{h}, b_{t+j}^{f}\right)_{j=1}^{d}$, that result from given choices by everyone else in the world economy. Next, we determine individual savings and consumption for the generation born in period t. We do this in the following manner:

$$c^i_{1,t} = \frac{\sum_{\tau=1}^{d} \frac{Z^i_{t+\tau}}{Z^i_{t+1}} \frac{w^i_{t+\tau} l^i_{\tau,t} + b^i_{t+\tau}}{\prod_{s=1}^{\tau} R^i_{t+s}}}{\sum_{\tau=1}^{d} \frac{\left(\frac{\beta^{\tau-1}\Psi^i_{\tau,t} \prod_{s=1}^{\tau} R^i_{t+s}}{R^i_{t+1}}\right)^{\frac{1}{\sigma}}}{\prod_{s=1}^{\tau} R^i_{t+s}}} \tag{17}$$

$$c^i_{\tau+1,t} = c^i_{\tau,t} \frac{Z_{t+\tau}}{Z_{t+\tau+1}} \left(\beta R^i_{t+\tau+1} \frac{\Psi^i_{\tau+1,t}}{\Psi^i_{\tau,t}}\right)^{\frac{1}{\sigma}} \tag{18}$$

$$s^i_{\tau,t} = w^i_{t+\tau} l^i_{\tau,t} + R^i_{t+\tau} s^i_{\tau-1,t} \frac{Z_{t+\tau-1}}{Z_{t+\tau}} + b^i_{t+\tau} - c^i_{\tau,t} \tag{19}$$

The allocation of capital is pinned down by the two goods-market clearing conditions. For country i this reads:

$$y^i_{t+1} + \sum_{j \in h,f} (1-\delta) \frac{x^{j,i}_t}{\pi^j_t} s^j_t \frac{Z^j_t L^j_t}{Z^i_{t+1} L^i_{t+1}} = \sum_{j \in h,f} \frac{x^{j,i}_{t+1}}{\pi^j_{t+1}} \left(c^j_{t+1} + s^j_{t+1}\right) \frac{Z^j_{t+1} L^j_{t+1}}{Z^i_{t+1} L^i_{t+1}} \tag{20}$$

Thus, we have three equations in only two unknowns. However, Walras' law ensures equilibrium in one market given equilibrium in the remaining markets. Therefore, we consider the goods-market clearing condition in ratio form when solving the model in practice. The iterative process over the entire truncated time horizon is then repeated over and over to update paths for factor prices, bequests, taxes, and benefits until convergence.

B. Steady State

This section characterizes the steady state. This is defined as a situation where the mortality pattern, population growth, and technology growth have all become constant, and growth-adjusted variables have subsequently converged. For guesses of the capital allocation, $\left(\kappa^h, \kappa^f\right)$, and bequests, $\left(b^h, b^f\right)$, one can directly determine the following aggregate variables:

$$y^i = A\left(\alpha\left(\kappa^i\right)^{\frac{\theta-1}{\theta}} + (1-\alpha)\right)^{\frac{\theta}{\theta-1}} \tag{21}$$

$$p = \left(\frac{\kappa^f y^h}{\kappa^h y^f}\right)^{\frac{1}{\theta}} \tag{22}$$

$$\pi^h = \left[\gamma_h\left(1 + p^{1-\eta}\frac{1-\gamma_h}{\gamma_h}\right)\right]^{\frac{1}{\eta-1}} \tag{23}$$

$$\pi^f = \left[\left(1-\gamma_f\right)\left(1 + p^{1-\eta}\frac{\gamma_f}{1-\gamma_f}\right)\right]^{\frac{1}{\eta-1}} \tag{24}$$

$$x^{h,h} = \frac{1}{1 + p^{1-\eta}\frac{1-\gamma_h}{\gamma_h}} \tag{25}$$

$$x^{h,f} = \frac{1}{p + p^{\eta}\frac{\gamma_h}{1-\gamma_h}} \tag{26}$$

$$x^{f,f} = \frac{1}{p + p^{\eta}\frac{1-\gamma_f}{\gamma_f}} \tag{27}$$

$$x^{f,h} = \frac{1}{1 + p^{1-\eta}\frac{\gamma_f}{1-\gamma_f}} \tag{28}$$

$$r = \alpha A^{\frac{\theta-1}{\theta}}\left(\frac{y^h}{\kappa^h}\right)^{\frac{1}{\theta}} - \delta \tag{29}$$

$$R = 1 + r \tag{30}$$

$$w^h = \pi^h (1-\alpha) A^{\frac{\theta-1}{\theta}}\left(y^h\right)^{\frac{1}{\theta}} \tag{31}$$

$$w^f = p\pi^f (1-\alpha) A^{\frac{\theta-1}{\theta}}\left(y^f\right)^{\frac{1}{\theta}} \tag{32}$$

Given all of these aggregates, we can determine consumption and savings over the entire life cycle by individual utility maximization:

$$c_1^i = \frac{\sum_{\tau=1}^{d} (1+g_z)^{\tau-1} \frac{w^i l_\tau^i + b^i}{R^\tau}}{\sum_{\tau=1}^{d} \frac{\left(\left(\beta R^i\right)^{\tau-1} \Psi_\tau^i\right)^{\frac{1}{\sigma}}}{R^\tau}} \tag{33}$$

$$c_\tau^i = \frac{c_1^i}{(1+g_z)^{\tau-1}} \left(\prod_{s=2}^{\tau} \beta R \psi_s^i\right)^{\frac{1}{\sigma}} \tag{34}$$

$$s_\tau^i = w^i l_\tau^i + \frac{R}{1+g_z} s_{\tau-1}^i + b^i - c_\tau^i \tag{35}$$

Given the optimal consumption-saving behavior of households, aggregate saving in each economy reads:

$$s^i = \frac{\sum_{\tau=1}^{d} \frac{\Psi_\tau^i}{(1+g_n)^\tau} s_\tau^i}{\sum_{\tau=1}^{r} \frac{\Psi_\tau^i}{(1+g_n)^\tau}} \tag{36}$$

Hence, recalling previous assumptions on the capital accumulation process, capital owned by country i reads:

$$k^i = \frac{1}{\pi^i} \frac{s^i}{(1+g_z)(1+g_n)} \tag{37}$$

Next, we solve numerically for guesses that are consistent with equations (38) through (40). First, capital-market clearing requires:

$$\kappa^h - k^h + \left(\kappa^f - k^f\right) \frac{Z_0^f n_0^f}{Z_0^h n_0^h} \frac{\sum_{\tau=1}^{r} \frac{\Psi_\tau^f}{(1+g_n)^\tau}}{\sum_{\tau=1}^{r} \frac{\Psi_\tau^h}{(1+g_n)^\tau}} = 0 \tag{38}$$

Similarly, goods-market clearing requires:

$$y^i \sum_{\tau=1}^{r} \frac{\Psi_\tau^i}{(1+g_n)^\tau} + \sum_{j \in h,f} \sum_{\tau=1}^{r} \frac{1-\delta}{(1+g_z)(1+g_n)} \frac{x^{j,i}}{\pi^j} \frac{Z_0^j n_0^j}{Z_0^i n_0^i} s_\tau^j \frac{\Psi_\tau^j}{(1+g_n)^\tau} = \sum_{j \in h,f} \sum_{\tau=1}^{d} \frac{\frac{x^{j,i}}{\pi^j} \Psi_\tau^j \left(c_\tau^j + s_\tau^j\right) \frac{Z_0^j n_0^j}{Z_0^i n_0^i}}{(1+g_n)^\tau} \tag{39}$$

For bequest, we require:

$$b^i = \frac{R}{(1+g_z)(1+g_n)} \frac{\sum_{\tau=1}^{d} \frac{\Psi_\tau^i - \Psi_{\tau+1}^i}{(1+g_n)^\tau} s_\tau^i}{\sum_{\tau=1}^{d} \frac{\Psi_\tau^i}{(1+g_n)^\tau}} \tag{40}$$

Thus, we can solve the steady state as a system of four equations in four unknowns. We do so using standard numerical solution methods.

B.1 Adjustment Periods

We consider MIT shocks to technology growth, fertility, and mortality. Thus, agents in some cohorts born before the shock may wish to revise their original plans for the future after learning new information. If new information arrives at the beginning of period t, an individual aged $a \in [1, d]$ (who was born in period $t-a$) solves:

$$\max_{\left\{C^i_{\tau,t-a}, S^i_{\tau,t-a}\right\}} U^i_{t-a} = \sum_{\tau=a}^{d} \beta^{\tau} \frac{\Psi^i_{\tau,t-a}}{\Psi^i_{a,t-a}} u\left(C^i_{\tau,t-a}\right)$$

$$s.t.$$

$$C^i_{\tau,t-a} + S^i_{\tau,t-a} = W^i_{t-a+\tau} l^i_{\tau,t-a} + R^i_{t-a+\tau} S^i_{\tau-1,t-a} + B^i_{t-a+\tau}$$

for given savings, $S^i_{a-1,t-a}$, carried over from the period before the shock. For given prices, this problem has a closed-form solution. After deriving the solution, we again make sure to growth adjust.

C. Figures and tables

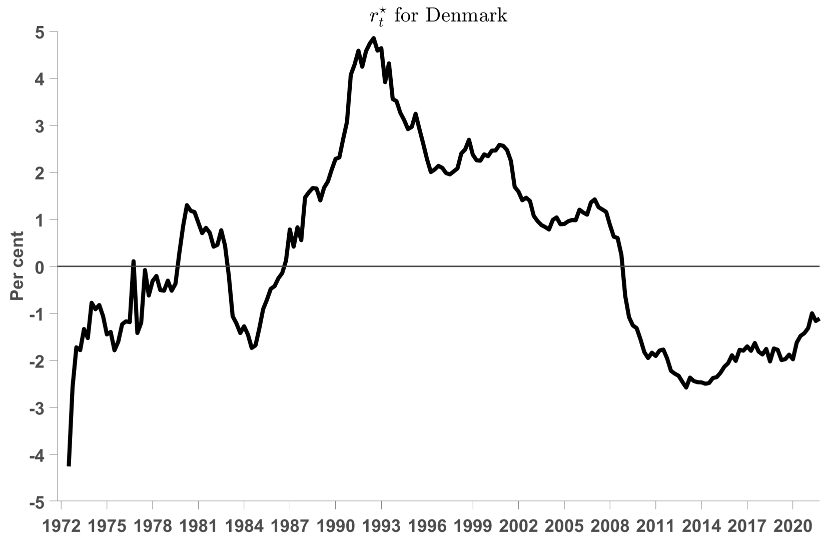

Figure 7: $r_t^\star$ **for Denmark**
In the figure is shown an estimate of $r_t^\star$ for Denmark, see Pedersen (2015).

PUBLICATIONS

NEWS

News offers quick and accessible insight into an Analysis, an Economic Memo, a Working Paper or a Report from Danmarks Nationalbank. News is published continuously.

ANALYSIS

Analyses from Danmarks Nationalbank focus on economic and financial matters. Some Analyses are published at regular intervals, e.g. *Outlook for the Danish economy* and *Financial stability*. Other Analyses are published continuously.

REPORT

Reports comprise recurring reports and reviews of the functioning of Danmarks Nationalbank and include, for instance, the *Annual report* and the annual publication *Danish government borrowing and debt*.

ECONOMIC MEMO

An Economic Memo is a cross between an Analysis and a Working Paper and often shows the ongoing study of the authors. The publication series is primarily aimed at professionals. Economic Memos are published continuously.

WORKING PAPER

Working Papers present research projects by economists in Danmarks Nationalbank and their associates. The series is primarily targeted at professionals and people with an interest in academia. Working Papers are published continuously.

DANMARKS NATIONALBANK
LANGELINIE ALLÉ 47
DK-2100 COPENHAGEN Ø
WWW.NATIONALBANKEN.DK

DANMARKS NATIONALBANK